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# Dobiński-type relations: some properties and physical applications

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**Abstract**

We introduce a generalization of the Dobiński relation through which we define a family of Bell-type numbers and polynomials. For all these sequences, we find the weight function of the moment problem and give their generating functions. We provide a physical motivation of this extension in the context of the boson normal ordering problem and its relation to an extension of the Kerr Hamiltonian.

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## 1. Introduction

In this paper, we consider the Dobiński relation and its generalization. This topic naturally belongs to the field of combinatorial analysis. The Dobiński relation [1] was first derived in connection with *Bell numbers* $B(n) = 1, 1, 2, 5, 52, 203, 877, \ldots, n = 0, 1, 2, \ldots,$ which describe partitions of a set [2, 3]. That remarkable formula represents the integer sequence $B(n)$ as an infinite sum of ratios

$$B(n) = \mathrm{e}^{-1}\sum_{k=0}^{\infty}\frac{k^n}{k!}. \tag{1}$$

Closely related to the Bell numbers are *Stirling numbers* of the second kind $S(n, k), k = 1, \ldots, n$, and the *Bell polynomials* defined as

$$B(n, x) = \sum_{k=1}^{n} S(n, k)x^k, \tag{2}$$

related to $B(n)$ by $B(n) = B(n, 1) = \sum_{k=1}^{n} S(n, k)$. For the Bell polynomials, the Dobiński relation (1) generalizes to

$$B(n, x) = \mathrm{e}^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!} x^k. \tag{3}$$

These formulae may be derived using either combinatorial or purely analytical methods starting from the original interpretation of Bell and Stirling numbers given in enumerative combinatorics [4, 5]. Accordingly, Stirling numbers $S(n, k)$ count the number of possible partitions of the $n$-element set into $k$ subsets (none of them empty) and Bell numbers $B(n)$ count all such partitions. We note that other pictorial representations can also be given, e.g., in terms of graphs [6] or rook numbers [7–9].

One may conversely take equation (1) (or equation (3)) as the definition of the Bell numbers (or polynomials). This observation suggests the generalization of these sequences through the Dobiński relation. In this paper, we introduce an extension of equation (1) and define the family of *Bell-type numbers* as

$$\mathcal{B}(n) = \sum_{k=0}^{\infty} \frac{[P(k)]^n}{D(k)}, \tag{4}$$

where $P(k)$ and $D(k)$ are any functions of $k = 0, 1, 2, \ldots$ such that $D(k) \neq 0$ and the above sum converges. Note that conventional Bell numbers are obtained for $P(k) = k$ and $D(k) = \mathrm{e}k!$.

This generalization was also pointed out in [10, 11] in connection with the log-normal distribution. Here, we focus on the general properties of our proposed definition and show that the very specific form of equation (4) results in a straightforward solution of the moment problem and calculation of the generating functions. We also comment on the connection to physics and interpret the sequences so defined in the context of the problem of the normal ordering of boson operators.

## 2. Generalized Dobiński relation and weight functions

Suppose that we want to solve the moment problem [12] for the sequence $\mathcal{B}(n)$, i.e. we seek a positive weight function $\mathcal{W}(y)$ such that $\mathcal{B}(n)$ is its $n$th moment

$$\mathcal{B}(n) = \int \mathrm{d}y\, y^n \mathcal{W}(y). \tag{5}$$

At this point, we do not specify the domain of $\mathcal{W}(y)$ or the limits of the integral. A closer look at equation (4) yields the following candidate for the weight function:

$$\mathcal{W}(y) = \sum_{k=0}^{\infty} \frac{\delta(y - P(k))}{D(k)}. \tag{6}$$

This is an infinite ensemble of weighted Dirac $\delta$ functions located at a specific set of points $\{P(k), k = 0, 1, 2, \ldots\}$ and is called a *Dirac comb*. If all the weights $1/D(k)$ are positive $(D(k) > 0)$ and normalized to 1 $\left(\sum_{k=0}^{\infty} 1/D(k) = 1\right)$ then equation (6) is a positive and normalized distribution which is a solution of the moment problem of equation (5). Whether it corresponds to the Hamburger, Stieltjes or Hausdorff moment problem depends on the range of the set $\{P(k), k = 0, 1, 2, \ldots\}$. For example, for the sequence of Bell numbers $B(n)$, the weight function $\mathcal{W}(y) = \mathrm{e}^{-1} \sum_{k=0}^{\infty} \frac{\delta(y-k)}{k!}$ is a positive and normalized distribution solving the Stieltjes moment problem, see figure 1. A solution of the Hamburger moment problem is generated by the set of *restricted* Bell numbers $B_{\bar{1}}(n) = 1, 0, 1, 1, 4, 11, 41, 162, \ldots$ for

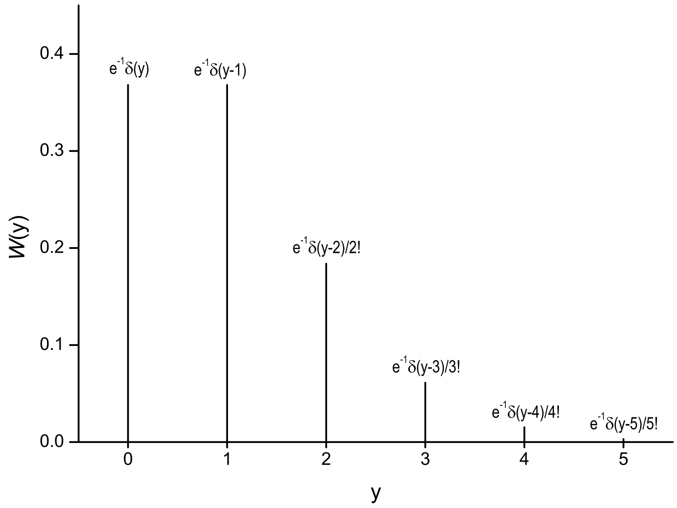


**Figure 1.** The portion for $0 \leqslant y \leqslant 5$ of the weight function $W(y) = \mathrm{e}^{-1}\sum_{k=0}^{\infty}\frac{\delta(y-k)}{k!}$ solving the Stieltjes moment problem for the Bell numbers $B(n)$. Height of the vertical lines is proportional to the weight of the Dirac $\delta$ functions.

$n = 0, 1, \ldots$ counting partitions without singletons [4]. They satisfy $B_{\bar{1}}(n) = \mathrm{e}^{-1}\sum_{k=0}^{\infty}\frac{(k-1)^n}{k!}$ with $P(k) = k-1$ and $D(k) = \mathrm{e}k!$; the measure is $\mathcal{W}_{\bar{1}}(y) = \mathrm{e}^{-1}\sum_{k=0}^{\infty}\frac{\delta(y-k+1)}{k!}$. On the other hand, the well-known Catalan numbers $C(n) = \frac{1}{n+1}\binom{2n}{n}$ are solutions of the Hausdorff moment problem [14].

The specific form of equation (4) simplifies the calculation of the generating functions. Taking the *exponential* generating function, substituting equation (4) and changing the summation order, one obtains

$$\mathcal{G}(\lambda) = \sum_{n=0}^{\infty}\mathcal{B}(n)\frac{\lambda^n}{n!} = \sum_{k=0}^{\infty}\frac{\mathrm{e}^{\lambda P(k)}}{D(k)}. \tag{7}$$

Evaluation of this series depends on the particular choice of the functions $P(k)$ and $D(k)$ and in general the series may be divergent. For the Bell and restricted Bell numbers, equation (7), it can be evaluated easily: $G(\lambda) = \sum_{n=0}^{\infty} B(n)\frac{\lambda^n}{n!} = \mathrm{e}^{\mathrm{e}^{\lambda}-1}$ and $G_{\bar{1}}(\lambda) = \sum_{n=0}^{\infty} B_{\bar{1}}(n)\frac{\lambda^n}{n!} = \mathrm{e}^{\mathrm{e}^{\lambda}-1-\lambda}$.

Similarly, for the *ordinary* generating function, one gets

$$\mathcal{G}_o(\lambda) = \sum_{n=0}^{\infty}\mathcal{B}(n)\lambda^n = \sum_{k=0}^{\infty}\frac{1}{D(k)\cdot(1-P(k)\lambda)}. \tag{8}$$

The same procedure can also be performed for other cases, e.g. for *hypergeometric* generating functions [13]. The choice of the denominator in the generating function may depend on $P(k)$, $D(k)$ and the purpose we need it for (e.g., when we need a convergent generating function for analytical calculations).

In the same manner, one could generalize equation (3) and define

$$\mathcal{B}(n, x) = \sum_{k=0}^{\infty}\frac{[P(k,x)]^n}{D(k,x)}. \tag{9}$$

The additional variable in the functions $P(n,x)$ and $D(n,x)$ does not pose any complication either in the proposed approach to the moment problem or in the evaluation of the generating functions. However, we must observe that in general $\mathcal{B}(n,x)$ has an infinite expansion in $x$ and only for particular choices of the functions $P(k,x)$ and $D(k,x)$ does it yield *polynomials*. This is certainly the case for $D(k,x) = k!\mathrm{e}^x x^{-k}$, and $P(k,x) = P(k)$ a polynomial in $k$. Therefore, we define the *Bell-type polynomials* as

$$\mathcal{B}(n,x) = \mathrm{e}^{-x}\sum_{k=0}^{\infty}\frac{[P(k)]^n}{k!}x^k. \tag{10}$$

As a result, the weight function of equation (6) takes the form

$$\mathcal{W}(x,y) = \mathrm{e}^{-x}\sum_{k=0}^{\infty}\frac{\delta(y-P(k))}{k!}x^k, \tag{11}$$

and the exponential generating function of equation (7) is

$$\mathcal{G}(\lambda,x) = \mathrm{e}^{-x}\sum_{k=0}^{\infty}\frac{\mathrm{e}^{\lambda P(k)}}{k!}x^k. \tag{12}$$

The case of conventional Bell polynomials is obtained for $P(k,x) = k$. Consequently, one gets the positive and normalized weight function $\mathcal{W}(y,x) = \mathrm{e}^{-x}\sum_{k=0}^{\infty}\frac{\delta(y-k)}{k!}x^k$ of the Stjelties moment problem, $B(n,x) = \int_0^\infty \mathrm{d}y\,y^n\mathcal{W}(y,x)$, and the exponential generating function $G(\lambda,x) = \sum_{n=0}^{\infty}B(n,x)\frac{\lambda^n}{n!} = \mathrm{e}^{x(\mathrm{e}^\lambda-1)}$ (see [15]). Analogous considerations can be applied to the polynomials generated by $B_{\bar{1}}(n)$ leading to $B_{\bar{1}}(n,x) = \mathrm{e}^{-x}\sum_{k=0}^{\infty}\frac{(k-1)^n}{k!}x^k$ and $G_{\bar{1}}(\lambda,x) = \sum_{n=0}^{\infty}B_{\bar{1}}(n,x)\frac{\lambda^n}{n!} = \mathrm{e}^{x(\mathrm{e}^\lambda-1-\lambda)}$.

## 3. Application to physics

Introduction of the generalized Bell-type numbers and polynomials through equations (4) and (10) is not merely a mathematical artifice but has a firm grounding in physics. We will show that they are related to the solution of the normal ordering problem for a general function of the number operator, with application to quantum partition functions and, as an example, a generalized Kerr-type Hamiltonian.

### *3.1. Normal ordering*

Consider the boson creation operator $a^\dagger$ and annihilation operator $a$ satisfying the commutator $[a,a^\dagger] = 1$. Suppose we are given a function of these operators. Its *normally ordered* form is obtained by moving all the creation operators to the left of the annihilation operators using the commutation relation. The normal ordering procedure is of fundamental importance in quantum mechanical calculations in the coherent state representation, the latter defined by the coherent states $|z\rangle = \mathrm{e}^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle$, where $a^\dagger a|n\rangle = n|n\rangle$, $\langle n|n'\rangle = \delta_{n,n'}$ and $a|z\rangle = z|z\rangle$ [16]. For example, if we take the $n$th power of the number operator $a^\dagger a$, the normal ordering procedure gives [17]

$$(a^\dagger a)^n = \sum_{k=1}^{n}S(n,k)(a^\dagger)^k a^k. \tag{13}$$

It involves Stirling numbers of the second kind $S(n,k)$ and the coherent state matrix element yields the Bell polynomial

$$\langle z|(a^\dagger a)^n|z\rangle = B(n,|z|^2). \tag{14}$$

Now, we consider a general polynomial of the number operator denoted by

$$\mathcal{H}_\alpha(a^\dagger a) = \sum_{k=N_0}^{N} \alpha_k (a^\dagger a)^k \tag{15}$$

with some constants $\alpha_k$; $N_0$ and $N$ are the smallest and largest indices of non-vanishing $\alpha_k$, respectively. Physically, $\mathcal{H}_\alpha$ may be thought of as a generalization of the Kerr Hamiltonian of quantum optics [18]. The $n$th power of $\mathcal{H}_\alpha$ defines the *Stirling-type numbers* as

$$[\mathcal{H}_\alpha(a^\dagger a)]^n = \sum_{k=N_0}^{nN} \mathcal{S}_\alpha(n,k)(a^\dagger)^k a^k, \tag{16}$$

and associated *Bell-type polynomials* (of order $nN$) are

$$\mathcal{B}_\alpha(n,x) = \sum_{k=N_0}^{nN} \mathcal{S}_\alpha(n,k) x^k. \tag{17}$$

We will now show that such polynomials defined in the normal ordering problem correspond to the Bell-type polynomials introduced in equation (10).

To this end, observe that $[a, a^\dagger] = [D, X] = 1$ where $D$ and $X$ are the derivative and multiplication operators. We first rewrite equation (16) in terms of $D$ and $X$:

$$[\mathcal{H}_\alpha(XD)]^n = \sum_{k=N_0}^{nN} \mathcal{S}_\alpha(n,k) X^k D^k. \tag{18}$$

By acting with the rhs of equation (18) on $\mathrm{e}^x$, one obtains $\mathrm{e}^x \mathcal{B}_\alpha(n,x)$. Action of the lhs on $\mathrm{e}^x$ is more involved. First, we apply it to the monomial $x^m$ which yields $[\mathcal{H}_\alpha(XD)]^n x^m = \left(\sum_{k=N_0}^{N} \alpha_k m^k\right)^n x^m$ from which $[\mathcal{H}_\alpha(XD)]^n\,\mathrm{e}^x = \sum_{m=0}^{\infty} \left(\sum_{k=N_0}^{N} \alpha_k m^k\right)^n \frac{x^m}{m!}$ follows. Combining these two observations, we deduce that

$$\mathcal{B}_\alpha(n,x) = \sum_{k=N_0}^{nN} \mathcal{S}_\alpha(n,k) x^k = \mathrm{e}^{-x} \sum_{k=0}^{\infty} \frac{[\mathcal{H}_\alpha(k)]^n}{k!} x^k, \tag{19}$$

which has the same form as equation (10) for $P(k) = \mathcal{H}_\alpha(k)$. The assumption that $\mathcal{H}_\alpha(x)$ is a polynomial guarantees that $\mathcal{B}_\alpha(n,x)$ is also a polynomial in $x$. Although this additional assumption may be irrelevant in general, as we have mentioned above it leads to infinite sequences of Stirling-type numbers.

### *3.2. Partition function integrand*

We have shown that the above approach gives an interpretation of the Bell-type polynomials and numbers in the context of the normal ordering problem. We now remark that the normally ordered exponential of a function of the number operator is the exponential generating function of the associated Bell-type polynomials. In the coherent state representation, it may be written as

$$\begin{aligned}\langle z|\mathrm{e}^{\lambda\mathcal{H}_\alpha(a^\dagger a)}|z\rangle = \mathcal{G}_\alpha(\lambda,|z|^2) &= \sum_{n=0}^{\infty} \mathcal{B}_\alpha(n,|z|^2)\frac{\lambda^n}{n!} \\ &= \mathrm{e}^{-|z|^2} \sum_{k=0}^{\infty} \frac{\mathrm{e}^{\lambda\mathcal{H}_\alpha(k)}}{k!} |z|^{2k},\end{aligned} \tag{20}$$

where in order to obtain the last equality we have used the Dobiński-type relation (19) and changed the summation order. Consequently, in view of the general properties of the coherent state representation [19], the normally ordered form of the exponential of $\mathcal{H}_\alpha(a^\dagger a)$ may be readily obtained as

$$\begin{aligned} e^{\lambda \mathcal{H}_\alpha(a^\dagger a)} &= :e^{-a^\dagger a} \sum_{k=0}^{\infty} \frac{e^{\lambda \mathcal{H}_\alpha(k)}}{k!} (a^\dagger a)^k: \\ &= \sum_{k=0}^{\infty} \frac{e^{\lambda \mathcal{H}_\alpha(k)}}{k!} (a^\dagger)^k |0\rangle\langle 0| a^k \end{aligned} \tag{21}$$

because of the relation $:e^{-a^\dagger a}: = |0\rangle\langle 0|$. Taking the matrix element of equation (21) between arbitrary states $|A\rangle$ and $|B\rangle$, we see that the Dobiński-type relations derived above are particular cases of the Fock space expansion

$$\langle A|\mathcal{H}_\alpha^n(a^\dagger a)|B\rangle = \sum_{k=0}^{\infty} \mathcal{H}_\alpha^n(k)\langle A|k\rangle\langle k|B\rangle. \tag{22}$$

This may serve as a basis for the investigation of quantum boson representations of classical combinatorial sequences, which we shall develop elsewhere. Here, we note that the last series in equation (20) is uniformly convergent for $|z|^2 \in [0, \infty)$ which means that $\mathcal{G}_\alpha(\lambda, |z|^2)$ may be given a well-defined analytical meaning and clear physical interpretation as the *partition function integrand* $\langle z|e^{-\beta\mathcal{H}}|z\rangle$ [20], (see also [21]), being in this case the Borel transform of the partition function. Since $\mathcal{G}_\alpha(\lambda, |z|^2)$ also generates the combinatorial sequence $\mathcal{B}_\alpha(n, |z|^2)$, we can say that the Dobiński-type relations derived above provide us with a method for constructing models of combinatorial field theories, such as those proposed a few years ago in [22]. This will inevitably relate quantum partition function expansions to combinatorial sequences, including, for example, combinatorial interpretations of the generalized Stirling and Bell numbers. Here, we simply recall our guiding example of the free Hamiltonian $\mathcal{H} = a^\dagger a$ intimately related through equation (13) to the conventional Stirling and Bell numbers counting partitions of a set. The restricted Bell polynomials mentioned throughout this paper constitute an example corresponding to $\mathcal{H} = a^\dagger a - 1$.

### *3.3. Generalized Kerr Hamiltonian: a combinatorial interpretation*

The last illustration we give is connected to an example of the generalized Kerr Hamiltonian in the interaction picture $\mathcal{H} = (a^\dagger)^M a^M$ where $M$ is a fixed positive integer. Equation (16) defines generalized Stirling numbers $\mathcal{S}_\alpha(n, k)$ which also have a transparent combinatorial interpretation. It comes down to considering $Mn$ distinguishable objects with anti-correlated $M$-set structure. This can be realized by colouring them in such a way that there are exactly $M$ objects of each colour, i.e. there are $n$ differently coloured $M$-sets. Now, the numbers $\mathcal{S}_\alpha(n, k)$ count the partitions of that set into $k$ subsets with the restriction that anti-correlated sets are divided among different subsets, i.e. all objects in each subset are of different colours. This means that we restrict partitions of an $Mn$ set by requiring anti-correlation of certain subsets. The Dobiński relation and the exponential generating function for the corresponding Bell polynomials may be easily read off from equations (19) and (20) substituting $\mathcal{H}_\alpha(k) = P(k) = k \cdot (k-1) \cdots (k-M+1)$. The discrete weight function for the generalized Bell numbers is concentrated on the infinite set $\{k \cdot (k-1) \cdots (k-M+1), k = M, M+1, \ldots\}$ and is given by $\mathcal{W}_\alpha(y) = \sum_{k=M}^{\infty} \delta(y - k \cdot (k-1) \cdots (k-M+1))/k!$.

## 4. Conclusion

In conclusion, we want to emphasize the advantages of the introduction of the Bell-type numbers and polynomials through the generalization of the Dobiński relation of equations (4) and (10). It enables a straightforward solution of the moment problem in the form of a Dirac comb, see equations (5), (6) and (11). Moreover, the calculation of the generating functions simplifies considerably in that framework (see equations (7), (8) and (12)). We have also pointed out that this generalization has immediate application to the boson normal ordering problem. We have interpreted a wide class of Stirling-type numbers as the expansion coefficients of normally ordered functions of the number operator. Further modifications of the structure of the infinite series of equations (4) and (10) may, in general, lead to moment problems with continuous weight functions and will be developed elsewhere.

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